Airyの浅水波が津波の正体である —波打際暴発のからくり—

Tunamis on a deep open sea and on a gentle sloping beach

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RIMS

Outline

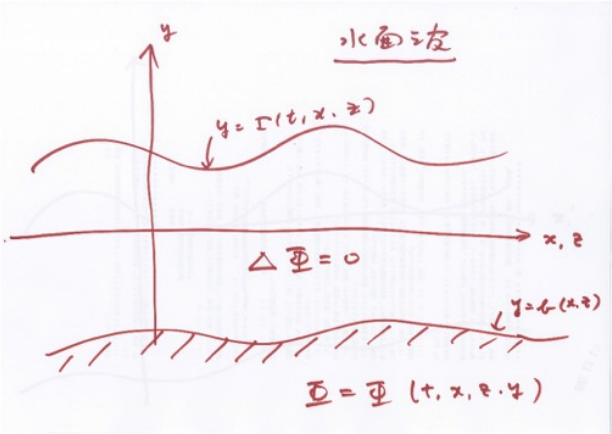
（イ）津波方程式提示、

（ロ）なぎさ（浅い水）での特異性顕現、

（ハ）その構造と原因の流体力学的解明、

（ニ）津波へ . . .

1. Water surface waves



2. Euler equations for 2-dim water surface waves

velocity potential $\Phi = \Phi(t, x)$: grad $\Phi = (u, v)$,
(u, v) : velocity vector field, wave-profile $y = \Gamma(t, x)$,
g : gravity, seabed : $y = b(x)$,

$$
\begin{aligned}
&(2.1) \quad \Phi_{xx} + \Phi_{yy} = 0, \\
&\qquad\qquad (x, y) \in \Omega_t = \{(x, y) \in \mathbb{R}^2, b(x) < y < \Gamma(t, x), t > 0\} \\
&(2.2) \quad -b_x \Phi_x + \Phi_y = 0, \ x \in \mathbb{R}^1, \ y = b(x) \\
&(2.3) \quad \Phi_t + \frac{1}{2}({\Phi_x}^2 + {\Phi_y}^2) + y = 0, \ x \in \mathbb{R}^1, y = \Gamma(t, x) \\
&(2.4) \quad \Gamma_t + \Gamma_x \Phi_x - \Phi_y = 0, \ x \in \mathbb{R}^1, \ y = \Gamma(t, x)
\end{aligned}
$$

initial data:

$$
\begin{aligned}
&\Phi(0, x, y) = \Phi_0(x, y) : \textit{harmonic} \\
&\Gamma(0, x) = \Gamma_0(x) > 0.
\end{aligned}
$$

2bis. Shallow water waves of finite amplitude

Shallow water waves such as:

$$\delta = \frac{h}{\lambda} = \frac{\text{“\textit{water depth}”}}{\text{“\textit{wave length}”}} \ll 1,$$
$$\varepsilon = \frac{a}{h} = \frac{\text{“\textit{amplitude}”}}{\text{“\textit{water depth}”}}, \ \textit{could be} \sim 1.$$

Among those shallow water waves, *long waves* are of the structure:

$$\left(\frac{\textit{mean water depth}}{\textit{wave length}}\right)^2 \quad \text{and} \quad \left(\frac{\textit{mean water amplitude}}{\textit{mean water depth}}\right)$$

are of the same order as infinitesimals when the wave length tends to infinity and the amplitude tends to zero.

(Dimensionless) Euler equations for water waves:

Defining the dimensionless variables by

$$(t,x,y)=\left(\frac{\lambda}{c}t',hy',\lambda x'\right),\quad (\Gamma,\Phi)=(h\Gamma',\ c\lambda\Phi'),$$

we have equations for water waves (by dropping prime sign):

$$\delta^2\Phi_{xx}+\Phi_{yy}=0,(x,y)\in\Omega_t=\{(x,y):x\in\mathbb{R},b(x)<y<\Gamma(t,x),t>0\}$$
$$-\delta^2 b_x\Phi_x+\Phi_y=0,x\in\mathbb{R}^1,y=b(x)$$
$$\delta^2(\Phi_t+\frac{1}{2}{\Phi_x}^2+y)+\frac{1}{2}{\Phi_y}^2=0,x\in\mathbb{R}^1,y=\Gamma(t,x)$$
$$\delta^2(\Gamma_t+\Gamma_x\Phi_x)-\Phi_y=0,x\in\mathbb{R}^1,y=\Gamma(t,x)$$

for the velocity potential Φ, wave profile function Γ with initial data $\quad \Phi(0,x,y)=\Phi_0(x,y),\quad \Gamma(0,x)=\Gamma_0(x)>0.$

Existence theorem: Levi-Civita, Struik, Lavrentiev, Nalimov, Ovsjannikov, Shinbrot, Kano-Nishida, Walter Craig, Yoshihara, Wu, Lannes

Shallow water wave equations of Airy:

the first nonlinear approximation of finite amplitude by the error of the order O(δ^2).
For $\{u = \Phi_x, \Gamma\}$ ($gravity = 1$):

$$u_t + uu_x + \Gamma_x = 0, \tag{2.5}$$

$$\Gamma_t + ((\Gamma - b(x))u)_x = 0. \tag{2.6}$$

3. Tunamis equations.

We give first **TUNAMIS EQUATION**.
Rewrite (2.5) and (2.6) as follows:

$$u_t + uu_x + (\Gamma - b(x))_x = -b_x, \tag{3.1}$$

$$(\Gamma - b(x))_t + ((\Gamma - b(x))u)_x = 0. \tag{3.2}$$

Let us now define γ by $\gamma^2 = \mathit{\Gamma} - b(x) > 0$,
$\gamma = \sqrt{\mathit{\Gamma} - b(x)} > 0$, we have then

$$P_t + (\gamma + u)P_x = -b_x, \tag{3.3}$$
$$Q_t - (\gamma - u)Q_x = -b_x \tag{3.4}$$

for $P = u + 2\gamma$ and $Q = u - 2\gamma$.

From these, we have finally the *TUNAMIS EQUATIONS*:

Definition 3.1 Tunamis equations. The following system of partial differential equations are **tunamis equations**:

$$P_t + \left(\gamma + u + \frac{b_x}{P_x}\right) P_x = 0 \tag{3.5}$$
$$Q_t - \left(\gamma - u - \frac{b_x}{Q_x}\right) Q_x = 0. \tag{3.6}$$

Let us discuss a little bit on this definition: water surface wave P, inland tunamis, propagate toward the beaches with the speed $\gamma + u + \dfrac{b_x}{P_x}$ modifying their velocity of cruising $\gamma + u = \sqrt{\Gamma - b(x)} + u$ by $\dfrac{b_x}{P_x}$ referring the state of sea-bed b_x in the connection with the structure P_x of himself. The same for the coastal tunamis Q. It is just from this structure of this "tunamis equations" start a violent tunamis development on a beach as we see later.

3bis. Tunamis or not tunamis.

Tunamis equations:

$$P_t + \frac{1}{4}(3P + Q)P_x = -b_x, \tag{3.7}$$

$$Q_t + \frac{1}{4}(P + 3Q)Q_x = -b_x, \tag{3.8}$$

with

$$\gamma + u = \frac{1}{4}(3P + Q), \quad \gamma - u = -\frac{1}{4}(P + 3Q).$$

No tunamis on a flat beach:

$$P_t + (\gamma + u)P_x = 0,$$
$$Q_t - (\gamma - u)Q_x = 0.$$

Tunamis: water waves on a beach with non vanishing b_x and $\Gamma - b$ becoming small.

4. Propagation speed of tunamis.

With b_x not identically vanishing, P propagate in the direction $x > 0$ with the speed

$$\gamma + u + \frac{b_x}{P_x} \tag{4.1}$$

that is, they satisfy the equation

$$P_t + (\gamma + u + \frac{b_x}{P_x})P_x = 0. \tag{4.2}$$

The propagation speed of P is affected by the third term in (4.1) being possibly $+\infty$ or $-\infty$ at the points where P_x vanishes: self-acceleration.

Problems:

(i) Does there really exist $x = X$ implying $P_x(X) = 0$ on shallow water near the coast, for example, in what situation does it occur?

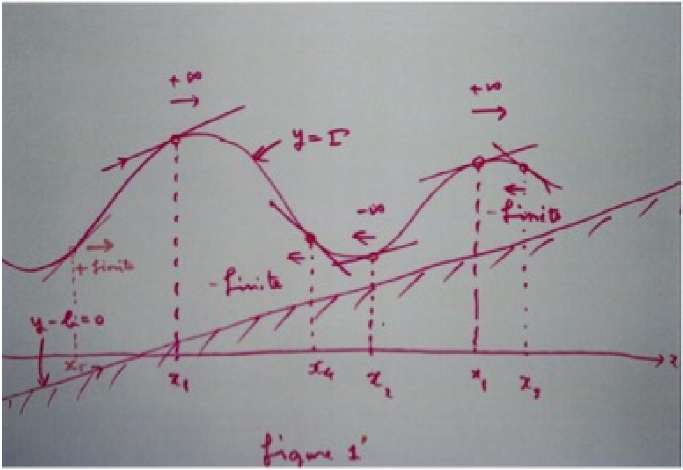
(ii) What would happen, then ?

5.1 Phenomenologically speaking:

[1] Under the condition $\gamma^2 = \mathit{\Gamma} - b(x) \ll 1$, on $x = X$, before the crest where $u_x < 0$, we have $P_x(t, X - \varepsilon) > P_x(t, X) = 0$ for $\varepsilon \ll 1$ and thus $P_x(t, X - \varepsilon) \to +0$ as $\varepsilon \to 0$, for tunamis $u_{xx} < 0$. It implies that our tunamis $P(t, x)$ get at $x = X - 0$ an instantaneous $+\infty$ propagation speed rushing thus as inland tunamis.

[2] Under the condition $\gamma^2 = \mathit{\Gamma} - b(x) \ll 1$ on $x = X$, after the trough where $u_x > 0$, we have $P_x(t, X - \varepsilon) < P_x(t, X) = 0$ for $\varepsilon \ll 1$ and thus $P_x(t, X - \varepsilon) \to -0$ as $\varepsilon \to 0$, for tunamis, "rather dynamic", $u_{xx} > 0$. It implies that our tunamis $P(t, x)$ get at $x = X - 0$ an instantaneous $-\infty$ propagation speed rushing consequently thus to the outer sea as offshore tunamis.

As an image for “sloping beach”:



5.2 The necessary conditions for P to realize $P_x(t, X) = 0$.

We can "find" $x = X$ mentioned above, before crest or after trough as follows:

Let first

$$x = X, P_x(t, X) = 0 \quad \text{where} \quad P_x = u_x + \frac{\Gamma_x - b_x}{\sqrt{\Gamma - b}} = 0.$$

Then we see

$$-u_x(X)\sqrt{\Gamma - b} = \Gamma_x - b_x \sim \pm 0$$

for $\gamma = \sqrt{\Gamma - b} \ll 1$.

Thus we see:

(i) near the crest where $u_x < 0$, we have

$$\Gamma_x(X) - b_x(X) \sim +0, \; \Gamma_x(X) - b_x(X) > 0.$$

(ii) near the trough where $u_x > 0$, we have

$$\Gamma_x(X) - b_x(X) \sim -0, \; \Gamma_x(X) - b_x(X) < 0.$$

6. Drawing by Hokusai:

An application: a possible alarm system/item.

If (these) two tangents satisfy conditions:
if the sea is shallow:

$$\gamma = \sqrt{\mathit{\Gamma} - b} \ll 1$$

we would have from $P_x(X) = 0$

$$-u_x(X)\sqrt{\mathit{\Gamma} - b(X)} = \mathit{\Gamma}_x(X) - b_x(X) \sim \pm 0.$$

And thus we see the situation would be dangerous. We should make a public alarm.